Кратчайшие Пути

① $w = 1$

② $w \geq 0$

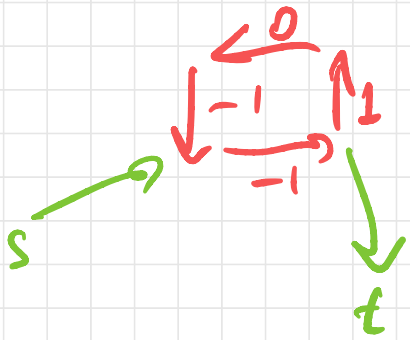
③ w отриц. Без отр. циклов

④ общий

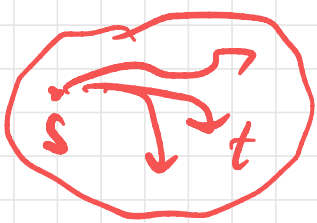
APSP / SSSP

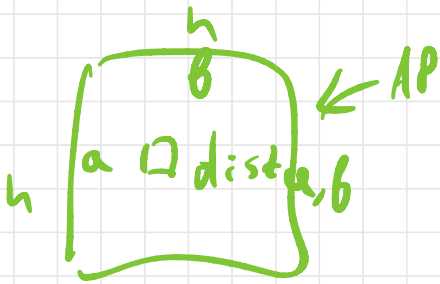
SS = Single Source

AP = All pairs



$\sum w_i < 0$

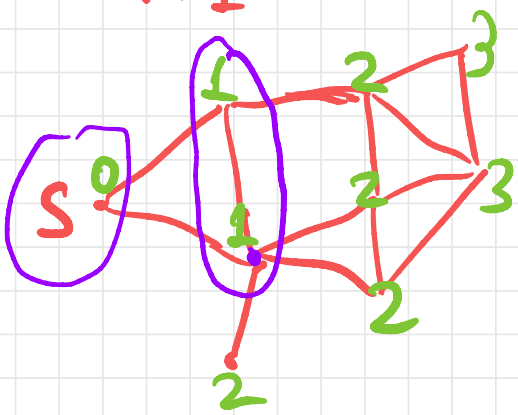




bfs
breadth-first search

Konrad Zuse, 1945
Edward Moore, 1959

- SSSP
- w = 1



$L_0 = \{s\}$

$L_1 = N(s)$

Упрощённый алгоритм

$L_k = \{v \mid dist(s, v) = k\}$

↑ будем их считать

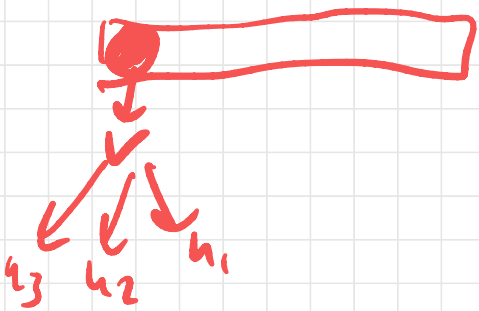
$L_0 = \{s\}$

for $d = 1 .. n-1$:

$L_d = N(L_{d-1}) \setminus (L_0 \cup ... \cup L_{d-1})$

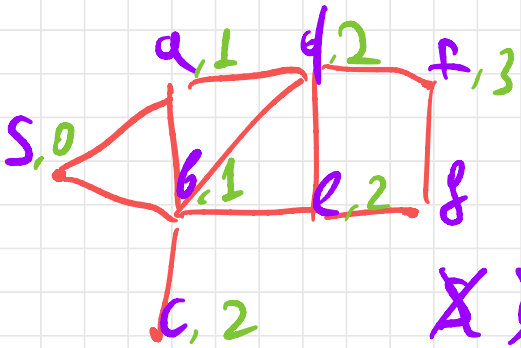
Поиск в ширину

Q = Queue

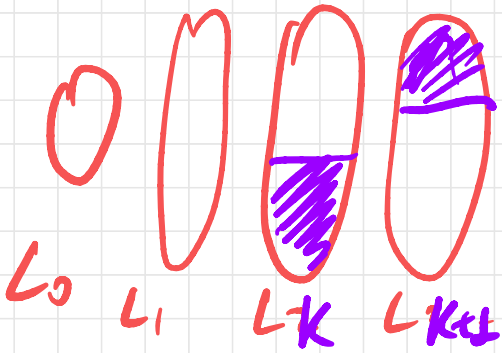


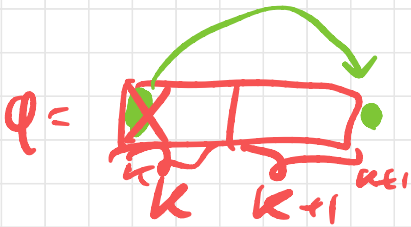
```
q = Queue()
q ← s
dist = [∞ .... ∞]
dist[s] = 0
while q ≠ ∅:
    v ← q
    for u in adj(v):
        if dist[u] == ∞:
            dist[u] = dist[v] + 1
            q ← u.
```

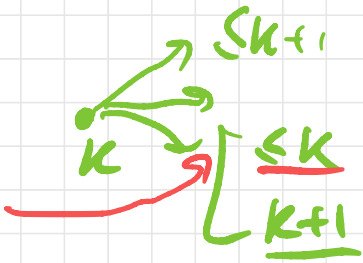
$O(V+E)$

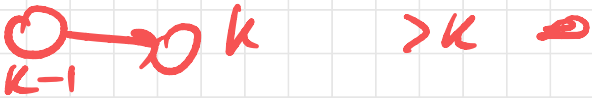


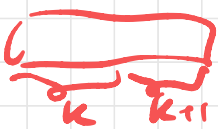
~~x~~ ~~x~~ ~~b~~ ~~d~~ e c f...
2 2 2 3









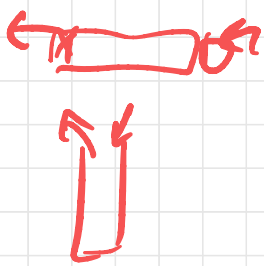
Утв 1: Очереди 

Утв 2: Все расст. посчитаны верно.

1) Неор граф / ор граф одинаковый ал.

2) q ← Queue()

q ← Stack()



Поиск в глубину

Рекурсивная реал. dfs: O(V) стек

q ← Stack() O(1) стек

3) dist[u]=k-1 dist[v]=k

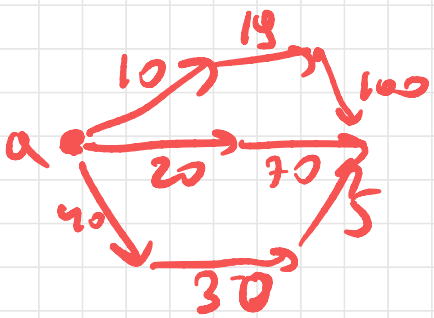
u → v

par[v]=u

Dijkstra

1956

- SSSP
- $w \geq 0$



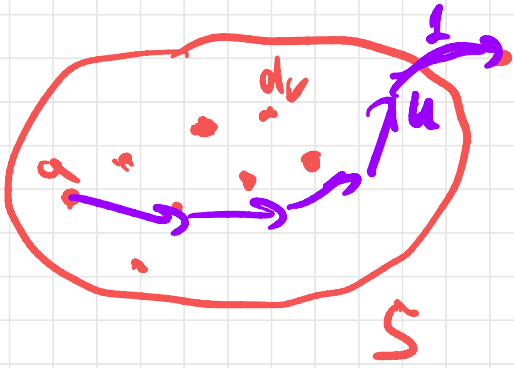
Алгоритм:

1) d_v = примерно расстояние от a до v

2) S^* – мн-во вершин, в которых мы знаем, что

$\forall v \in S: \quad d_v = dist(a, v)$

3)



d_v = кратч. путь такого вида

$\forall v \notin S \quad d_v = \min\limits_{\substack{u \in in(v) \\ u \in S}} d_u + w_{u,v}$

Алгоритм

1) $S = \{a\}$, $d_a = 0$
   for $(v, w) \in adj(a)$
   $d_v = w$

2) $|S| \to |S| + 1$

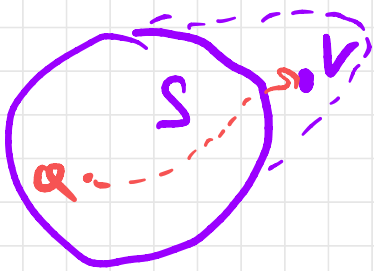
повторим $n-1$ раз

$S = V(G)$

Переход:

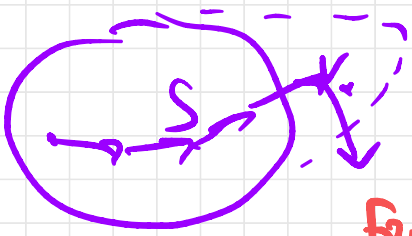
Утв: $V \notin S$, и из таких вершин $d_v \to \min$,

Тогда $d_v = dist(a, v)$



$S \to S \cup \{v\}$

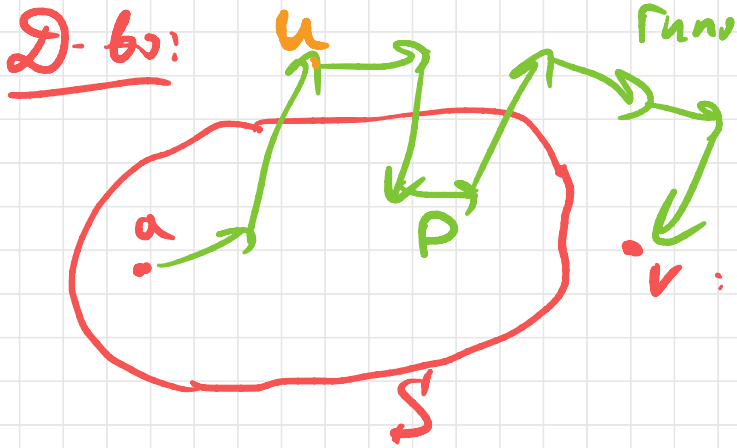
$S \to S \cup \{v\}$



For $(u, w) \in adj(v)$

$d_u = \min(d_u, d_v + w)$ ← d_u min = $d_v + w$

Д-во:



Гипотез. опт. путь.

v : d_v → min

$d_v \neq dist(a,v)$

P – мин. кратч. путь ($dist(a,v)$)

u – первая вершина вне S

$dist(P) = dist(a,v)$

$dist(P_1 \dots P_k) + dist(P_k \dots P_{|P|})$

$P_1 = v$, $P_k = u$, $P_{|P|} = v$

d_u, 0

$dist(P) \geq d_u \geq d_v$

Реализуем

d = [+∞ ... +∞]

used = [0 ... 0]

$v \in S$ ⇒ used[v] = 1

d[a] = 0

for i = 1 .. n:

  v: !used[v] and d[v] → min — O(n)

  used[v] = 1

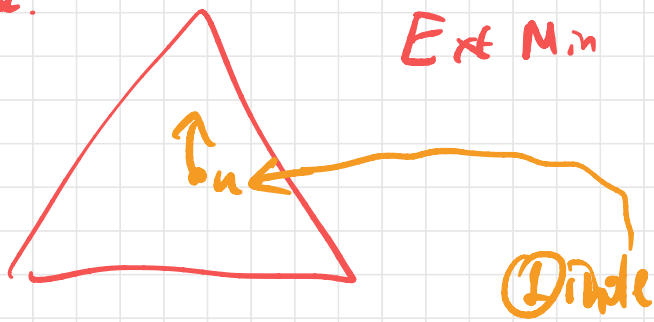
  for (u, w) in adj[v]:

    d[u] = min(d[u], d[v] + w)

$O(V^2 + E)$

Должно, что $E << V^2$

Куча:



v: !used[v] and d[v] → min

↓

Ext Min

d[u] - min = d[v] + w

↓

① DecreaseKey (SiftUp)

② Добавить ещё раз

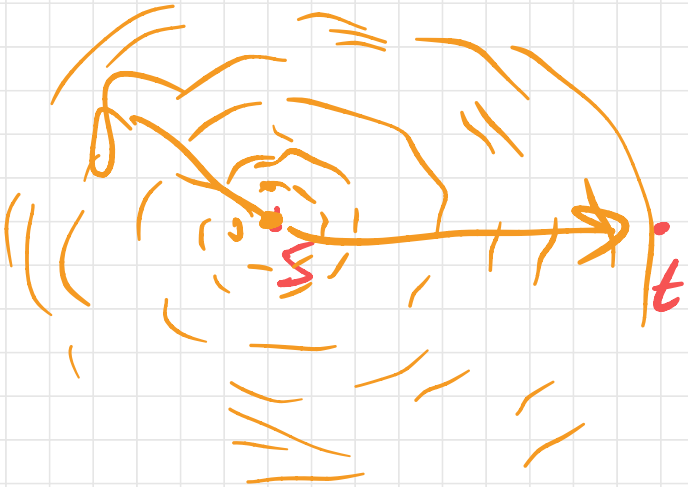
Реализация через (2)

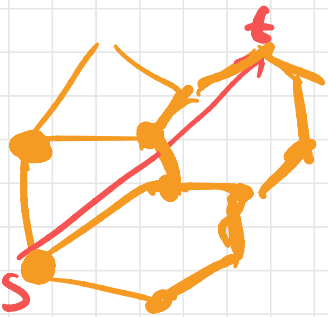
```
H = heap()
d = [∞ ... ∞]
d[a] = 0
H.push((d[a], a))

while H ≠ ∅:
    dv, v = H.pop()
    if dv > d[v]:
        return
    for u, w in adj[v]:
        if d[v] + w < d[u]
            d[u] = d[v] + w
            H.push((d[u], u))
```

$O(E \log V)$

	Overview	Алгоритмы	
BFS	SSSP	$W=1$	$O(V+E)$
dijkstra	SSSP	$W \geq 0$	$O(V^2+E)$
			$O(E \log V)$
			$O(E+V \log V)$
Bellman-Ford	SSSP	W любой	$O(VE)$
floyd-Warshall	APSP	W любые	$O(V^3)$
Johnson's	APSP	Нет отр цикла	$O(V^2 \log V + VE)$
Thorup	SSSP	$W \geq 0$ неор	$O(V+E)$
Thorup 2004	SSSP	$W \geq 0$	$O(E+V \log\log V)$

A* (A star)

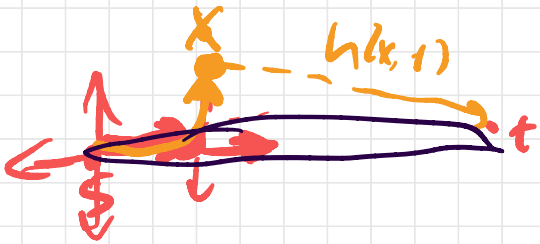




$h(V)$ = оценка расстояния от V до t.

Dijkstra:
$V: d_V \to min$

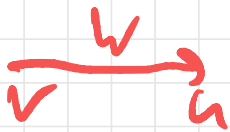
A^*:
$V: d_V + h(V) \to min$



Утв 1: $h(v) \leq dist(v,t)$, то
$(h(t)=0)$

$\Rightarrow$ найдём верный путь

Утв 2:



$h(v) \leq h(u) + w$ – консист

То A* эквив. алг. дейкстры для

$w_{v,u} \rightarrow w'_{v,u} =$
$= w_{v,u} + h_u - h_v$
≥ 0

$\Rightarrow$ Ни какую вершину не стоит рвать

$$w'_{s,v_1} + w'_{v_1,v_2} + w'_{v_2,t}$$

$$w_{s,v_1} + w_{v_1,v_2} + w_{v_2,t}$$

$$h_{v_1} - h_s$$

$$h_{v_2} - h_{v_1}$$

$$h_t - h_{v_2}$$

$$w'(P) = w(P) + \underline{h_t - h_s}$$

У.в: кр. путь в w'

эквив

кр. путь в w

Утв: дейкстра в w':

$d'_v \to \min$

v: $dist(p_1 \dots p_{|p|}) + h_v - h_s$

s — v — $\min$

$d'_v = d_v + h_v - h_s$